

## Background

- Learning algorithm  $\mathcal{A} : S \rightarrow W$  i.e. mapping a training sample to a hypothesis.
- Gen. Err. =  $\mathbb{E}[\text{Test Err.} - \text{Train Err.}] \leq \text{Gen. Bound.}$
- Information-theoretic (IT) bounds belong to the class of Gen. Bound.

## Limitations of IT Generalization bounds

- Original input-output mutual information (IOMI) (e.g.,  $I(W; S)$  in [5]) based bound can  $\rightarrow \infty$  😞.
- $\Rightarrow$  solved by conditional mutual information (CMI)  $I(W; U|\tilde{Z})$  in [4] 😊.
- Slow convergence rate, e.g.,  $\mathcal{O}(1/\sqrt{n})$  😞  $\Rightarrow$  mitigated by [3, 6] and so on 😊.
- Non-vanishing in Stochastic Convex Optimization (SCO) problems [2] 😞!

## Contributions

Our contribution: Incorporating stability-based analysis into IT framework which improves both stability-based bounds and IT bounds.

## Key Observation from Algorithmic Stability

- Given  $S = \{Z_i\}_{i=1}^n$  and  $Z'_i$ :
 
$$Z_1, \dots, Z_i, \dots, Z_n \xrightarrow{\mathcal{A}} W \Rightarrow \text{Loss of } (W, Z)$$

$$Z_1, \dots, Z'_i, \dots, Z_n \xrightarrow{\mathcal{A}} W^{-i} \Rightarrow \text{Loss of } (W^{-i}, Z)$$
- $\mathcal{A}$  is Stable  $\iff$  Loss of  $(W^{-i}, Z)$  is close to Loss of  $(W, Z)$ .
  - Uniform Stability [1]:  $\sup_{W, W^{-i}, Z} |\text{Loss of } (W, Z) - \text{Loss of } (W^{-i}, Z)| \leq \text{Unif. Stability Param.}$
  - Sample-Conditioned Hypothesis (SCH) Stability in this paper:  $\mathbb{E}_{W, W^{-i}} [\sup_Z |\text{Loss of } (W, Z) - \text{Loss of } (W^{-i}, Z)|] \leq \text{SCH Stability Param.}$ , where  $Z$  can be either  $Z_i$  or  $Z'_i$ .

Some terminologies

- Evaluated Data  $Z \in (Z_i, Z'_i)$ ;
- (Neighboring) Hypothesis pair:  $(W, W^{-i})$
- Membership: e.g.  $\mathbb{1}\{\text{Evaluated Data} = Z_i\}$

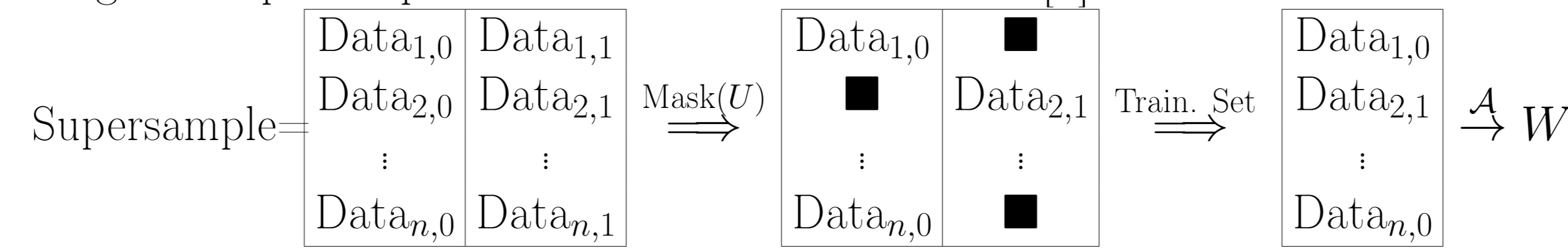
## Main Theorem (informal.)

If  $\mathcal{A}$  is stable, then  
 Gen. Err.  $\lesssim \text{Stability Param.} \times \underbrace{\sqrt{I(\text{Evaluated Data}; \text{Membership} | \text{Hypothesis Pair})}}_{\text{New CMI}}$

Generalization, in this context, pertains to the ability to infer, given  $(W, W^{-i})$  and Evaluated Data, whether the Evaluated Data corresponds to  $Z_i$  or  $Z'_i$ .

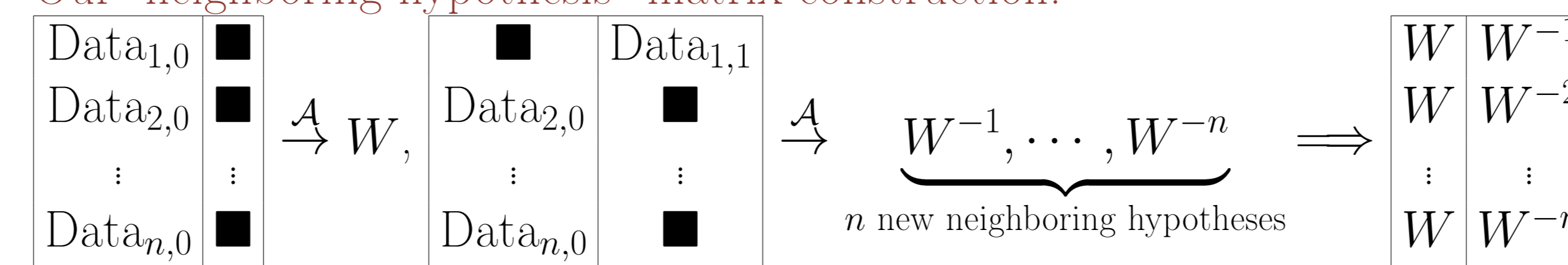
## Novel Construction: "Neighboring-Hypothesis" Matrix

- Original "supersample" matrix construction in CMI [4]:

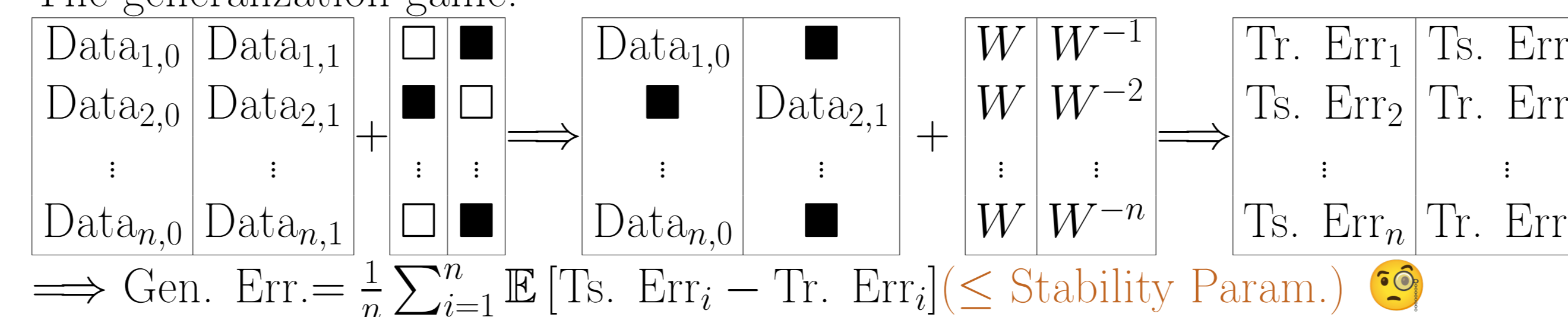


$\xrightarrow{\text{CMI}} I(W; U | \text{Supersample})$ : membership inference of train. set.

- Our "neighboring-hypothesis" matrix construction:

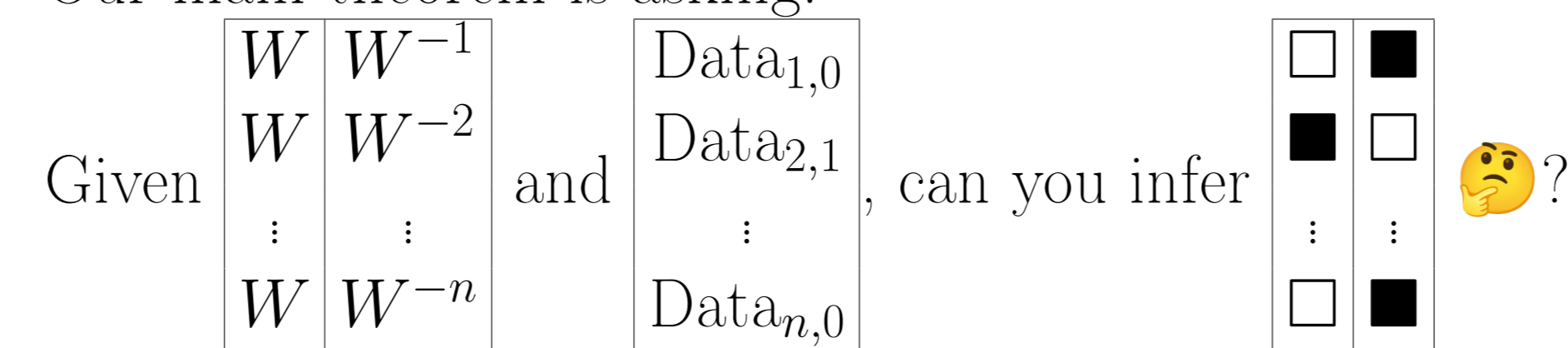


- The generalization game:



$\Rightarrow \text{Gen. Err.} = \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\text{Ts. Err}_i - \text{Tr. Err}_i] (\leq \text{Stability Param.})$  😊

- Our main theorem is asking:



## More Technical? All about Bounding CGF

Recall Donsker-Varadhan (DV) lemma:

$$\text{Gen. Err.} \leq \inf_{t>0} \frac{\text{IOMI or CMI} + \text{CGF}}{t}$$

Let  $f_{\text{DV}}$  be so-called DV auxiliary function, then

$$\text{CGF} = \log \mathbb{E}[\exp(t \cdot f_{\text{DV}})] \leq \text{Some Concentration Bound.}$$

- Typical choices of  $f_{\text{DV}}$  in previous works:

$$f_{\text{DV}} = \begin{cases} \text{Single Loss of } (w, z') \\ \text{Loss of } (w, z') - \text{Expected Loss of } (w, Z') \\ \text{Loss of Data 1} - \text{Loss of Data 2 for the same } w \end{cases}$$

where in the last choose, Data 1 is chosen uniformly from a data pair, e.g.,  $(Z^0, Z^1)$ , decided by a  $U \sim \text{Bern}(1/2) \Rightarrow \text{Data 1} = Z^U, \text{Data 2} = Z^{1-U}$ .

- In this paper:

$$f_{\text{DV}} = \begin{cases} \text{Loss of } (w, z') - \text{Conditional Expected Loss of } (W^{-i}, z') \\ \text{Loss of Hypothesis 1} - \text{Loss of Hypothesis 2 for the same } z' \end{cases}$$

where in the last choose, Hypothesis 1 is chosen uniformly from a neighboring hypothesis pair, e.g.,  $(W^0, W^1)$ , decided by a  $U \sim \text{Bern}(1/2) \Rightarrow \text{Hypothesis 1} = W^U, \text{Hypothesis 2} = W^{1-U}$ .

## Application: Stochastic Convex Optimization Problems

SCO setting: Hypothesis set is convex; Objective function is convex.

- In convex-Lipschitz-bounded (CLB) counterexamples (which is a subset of SCO problems) given by [2]:

$$\text{Gen. Err.} \leq \mathcal{O}(1/\sqrt{n}).$$

- Previous IOMI or CMI bound:  $\mathcal{O}\left(\alpha \sqrt{\frac{\text{IOMI or CMI}}{n}}\right)$ , where  $\alpha$  usually satisfies

that CGF  $\leq \frac{t^2 \alpha^2}{2}$ .

e.g.,  $\alpha$  can be a SubGaussian variance proxy or

$$\alpha = \sup_{\text{Hypothesis, Data Pair}} |\text{Loss of Data 1} - \text{Loss of Data 2}|.$$

- [2] shows that

$\alpha = \mathcal{O}(1)$  (=Lip. Param.  $\times$  Diam. of Hypothesis Domain)

and Previous IOMI  $\geq$  Previous CMI =  $\mathcal{O}(n)$ .

$\Rightarrow \mathcal{O}\left(\alpha \sqrt{\frac{\text{IOMI or CMI}}{n}}\right) \in \mathcal{O}(1) \Rightarrow$  Fail to explain the learnability 😞.

- Our new CMI bound:

Stability Param. =  $\mathcal{O}(1/\sqrt{n})$

and New CMI =  $\mathcal{O}(1)$ .

$\Rightarrow$  New CMI Bound  $\in \mathcal{O}(1/\sqrt{n}) \Rightarrow$  Can explain the learnability 😊!

- Wait, Stability Param. itself can serve as a generalization bound, why do we need IOMI or CMI 😞?

There is another CLB example in our paper where Stability Param. =  $\mathcal{O}(1/\sqrt{n})$  but Gen. Err.  $\leq$  New CMI Bound =  $\mathcal{O}(1/n)$  😊 Check it!

## Concluding Remarks

- Take-Home Message: Selecting the Suitable DV Auxiliary Function for Varied Problem Contexts.
- There are additional choices for SCH stability, allowing us to establish connections with the Bernstein condition or achieve faster-rate bounds in certain cases.
- Our new CMI maintains the same expressiveness as the original CMI and preserves its boundedness property. The comparison between the new CMI and the original CMI in a broader context remains an open question.

## References

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