CST 3526 Stochastic Process	Lecture 11 - 11/27/2025

Lecture 11: Second-order Processes I

Lecturer: Ziqiao Wang

In this lecture, we will introduce second-order processes, with a particular focus on stationary processes, including strict-sense and wide-sense stationarity. We will also briefly discuss a representative non-stationary process—the cyclostationary process and its wide-sense counterpart.

1 Partial Characterizations of a Random Process

Recall that the *mean function* of a random process $X(t)$ is defined as

$$m_X(t) \triangleq \mathbb{E}\left[X_t\right] = \int_{-\infty}^{\infty} x\, dF_{X_t}(x) = \int_{-\infty}^{\infty} x f_{X_t}(x)\, dx, \qquad t \in \mathcal{T}. \tag{1}$$

Since this can be obtained from the first–order distributions alone, the mean function is called a *first–order statistic* of the process. In addition, recall the *autocorrelation function* of $X(t)$ is

$$R_X(t_1, t_2) \triangleq \mathbb{E}\left[X_{t_1} X_{t_2}\right], \qquad t_1, t_2 \in \mathcal{T}, \tag{2}$$

and the *autocovariance function*

$$C_X(t_1, t_2) \triangleq \mathrm{Cov}(X_{t_1}, X_{t_2}) = R_X(t_1, t_2) - m_X(t_1)\, m_X(t_2), \qquad t_1, t_2 \in \mathcal{T}. \tag{3}$$

Because $R_X(t_1, t_2)$ and $C_X(t_1, t_2)$ depend on the second–order distributions, but not only on the first–order ones, they are referred to as *second–order statistics* of the process. Note that the variance of X_t is

$$\mathrm{Var}(X_t) = C_X(t, t) = R_X(t, t) - |m_X(t)|^2.$$

The (instantaneous) mean power of the process is given by

$$\mathbb{E}\left[|X_t|^2\right] = R_X(t, t), \qquad t \in \mathcal{T}. \tag{4}$$

Hence, the variance function $\mathrm{Var}(X_t)$ and the mean power function can be computed from the first–order distributions of the process and are thus also first–order statistics.

When the mean and autocorrelation functions do exist and are finite, we say the random process is a second-order process.

2 Stationary Processes

Suppose a random process $X(t)$ defined over "time", where $\mathcal{T}$ may be continuous ($\mathcal{T} = \mathbb{R}$) or discrete ($\mathcal{T} = \mathbb{Z}$). In many applications, the statistical properties of the process do not change if the time origin is shifted. That is, if we observe the process or a time-shifted version of it, the two should be statistically indistinguishable.

2.1 Strict-Sense Stationary (SSS) Random Process

For this condition to hold, we require that all finite-dimensional distributions be invariant under time shifts.

Definition 2.1. A discrete-time or continuous-time random process $X(t)$ is stationary if the joint distribution of any set of samples does not depend on the placement of the time origin.

That is, for any $t_1, \ldots, t_n, t, \tau \in \mathcal{T}$ and any real values $x_1, \ldots, x_n$,

$$F_{X_{t_1}, X_{t_2}, \cdots, X_{t_n}}(x_1, x_2, \ldots, x_n) = F_{X_{t_1+\tau}, X_{t_2+\tau}, \cdots, X_{t_n+\tau}}(x_1, x_2, \ldots, x_n). \tag{5}$$

When (5) holds, the process is called a **strict-sense stationary (SSS)** random process.

If the process is *second-order stationary* (i.e., has finite first and second moments), then stationarity implies constraints on the mean and autocorrelation functions.

Let $X(t)$ be such a process. Then for all $t, \tau \in \mathcal{T}$,

$$m_X(t) \triangleq \mathbb{E}[X_t] = \mathbb{E}[X_{t+\tau}] = m_X(t+\tau). \tag{6}$$

Thus, the mean function must be constant:

$$m_X(t) = \text{constant} \triangleq m_X. \tag{7}$$

Next, consider the autocorrelation function:

$$\begin{aligned} R_X(t_1, t_2) &\triangleq \mathbb{E}[X_{t_1} X_{t_2}] && (8) \\ &= \mathbb{E}[X_{t_1+\tau} X_{t_2+\tau}], \qquad \forall\, t_1, t_2, \tau \in \mathcal{T}. && (9) \end{aligned}$$

Setting $\tau = -t_1$ yields:

$$R_X(t_1, t_2) = R_X(0, t_2 - t_1). \tag{10}$$

Since it depends only on the time difference, define

$$R_X(\tau) \triangleq R_X(t, t+\tau), \qquad \tau \in \mathcal{T}. \tag{11}$$

Similarly, the autocovariance function satisfies:

$$C_X(t_1, t_2) = C_X(0, t_2 - t_1), \tag{12}$$

which motivates the definition:

$$C_X(\tau) \triangleq \mathbb{E}[(X_t - m_X)(X_{t+\tau} - m_X)] = C_X(t, t+\tau), \qquad \tau \in \mathcal{T}. \tag{13}$$

We have therefore shown that a second-order stationary process satisfies:

$$\begin{aligned} &\text{(a)} \quad m_X(t) \text{ is independent of } t, && (14) \\ &\text{(b)} \quad R_X(t_1, t_2) \text{ depends only on } t_2 - t_1. && (15) \end{aligned}$$

2.2 Wide-Sense Stationary (WSS) Random Process

The converse of the above is not necessarily true, which motivates a more general definition.

Definition 2.2. A random process $X(t)$ (with $\mathcal{T} = \mathbb{R}$ or $\mathcal{T} = \mathbb{Z}$) is said to be *wide-sense stationary (WSS)* if:

$$\begin{aligned} m_X(t_1) &= m_X(t_2), && (16) \\ R_X(t_1, t_2) &= R_X(0, t_2 - t_1), && (17) \end{aligned}$$

for all $t_1, t_2 \in \mathcal{T}$.

From the previous results, for a WSS process, the autocovariance function satisfies

$$C_X(t_1, t_2) = C_X(0, t_2 - t_1),$$

and hence depends only on the time difference $(t_2 - t_1)$. When referring to the mean, autocorrelation, and autocovariance of a WSS process, we usually denote the constant mean by m_X, the autocorrelation by $R_X(\tau)$ as defined in (11), and the autocovariance by $C_X(\tau)$ as defined in (13).

Although second-order stationarity implies the form $C_X(t_1, t_2) = C_X(t_2 - t_1)$, a process may fail to be WSS if the mean does not exist or is not constant, or if the autocorrelation is unbounded. **A stationary process is also wide-sense stationary if and only if it is a second-order process**. For a WSS second-order process, we must have

$$C_X(t_1, t_2) = C_X(t_2 - t_1).$$

The following example shows that some wide-sense stationary processes are not stationary.

Example 1. Let X_n consist of two interleaved sequences of independent random variables. For n even, X_n assumes the values ± 1 with probability $1/2$; for n odd, X_n assumes the values $1/3$ and -3 with probabilities $9/10$ and $1/10$, respectively.

X_n is not stationary since its pmf varies with n. It is easy to show that X_n has mean

$$m_X(n) = 0 \qquad \text{for all } n$$

and covariance function

$$C_X(i, j) = \begin{cases} \mathbb{E}[X_i]\mathbb{E}[X_j] = 0, & i \neq j, \\ \mathbb{E}[X_i^2] = 1, & i = j. \end{cases}$$

X_n is therefore wide-sense stationary.

Properties of a WSS process We now develop several results that enable us to deduce properties of a WSS process from properties of its autocorrelation function.

First, the autocorrelation function at $\tau = 0$ gives the average power (second moment) of the process:

$$R_X(0) = \mathbb{E}[X(t)^2] \qquad \text{for all } t. \tag{18}$$

Second, the autocorrelation function is an even function of τ since

$$R_X(\tau) = \mathbb{E}[X(t+\tau)X(t)] = \mathbb{E}[X(t)X(t+\tau)] = R_X(-\tau). \tag{19}$$

Third, the autocorrelation function is a measure of the rate of change of a random process in the following sense. Consider the change in the process from time t to $t + \tau$:

$$\begin{aligned} P\big(|X(t+\tau) - X(t)| > \varepsilon\big) &= P\big((X(t+\tau) - X(t))^2 > \varepsilon^2\big) \\ &\leq \frac{\mathbb{E}[(X(t+\tau) - X(t))^2]}{\varepsilon^2} \\ &= \frac{2\{R_X(0) - R_X(\tau)\}}{\varepsilon^2}. \end{aligned}$$

where we used the Markov inequality, to obtain the upper bound. The inequality above states that if $R_X(0) - R_X(\tau)$ is small, that is, $R_X(\tau)$ drops off slowly, then the probability of a large change in $X(t)$ in τ seconds is small.

Fourth, the autocorrelation function is maximum at $\tau = 0$. We use the Cauchy–Schwarz inequality:

$$|\mathbb{E}[XY]|^2 \leq \mathbb{E}[X^2]\mathbb{E}[Y^2], \tag{9.67}$$

for any two random variables X and Y. If we apply this equation to $X(t+\tau)$ and $X(t)$, we obtain

$$R_X(\tau)^2 = \mathbb{E}^2[X(t+\tau)X(t)] \leq \mathbb{E}[X(t+\tau)^2]\mathbb{E}[X(t)^2] = R_X(0)^2.$$

Thus

$$|R_X(\tau)| \leq R_X(0). \tag{9.68}$$

Fifth, if $R_X(0) = R_X(d)$, then $R_X(\tau)$ is periodic with period d and $X(t)$ is mean square periodic, that is, $\mathbb{E}[(X(t+d) - X(t))^2] = 0$. Notice that

$$\mathbb{E}^2[(X(t+\tau+d) - X(t+\tau))X(t)] \leq \mathbb{E}[(X(t+\tau+d) - X(t+\tau))^2]\,\mathbb{E}[X(t)^2],$$

which implies that

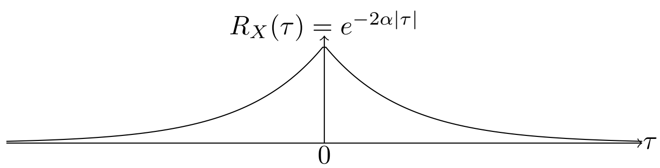
$$\{R_X(\tau+d) - R_X(\tau)\}^2 \leq 2\{R_X(0) - R_X(d)\}R_X(0). \tag{20}$$

Thus $R_X(d) = R_X(0)$ implies that the right-hand side of the equation is zero, and thus $R_X(\tau+d) = R_X(\tau)$ for all τ. Repeated applications of this result imply that $R_X(\tau)$ is periodic with period d.

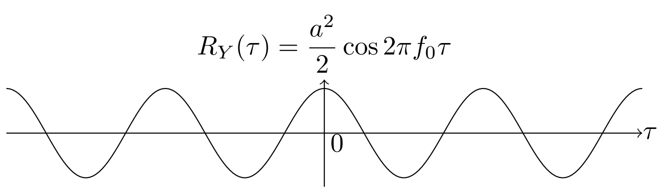
The fact that $X(t)$ is mean square periodic follows from

$$\mathbb{E}[(X(t+d) - X(t))^2] = 2\{R_X(0) - R_X(d)\} = 0. \tag{21}$$

Figure (a-b) show two autocorrelation functions for the WSS process.



(a)



(b)

3 Cyclostationarity and Wide-Sense Cyclostationarity

Another important form of stationarity is based on invariance under shifts that occur at integer multiples of a positive constant T_0.

Definition 3.1. A discrete-time or continuous-time random process $X(t)$ is said to be cyclostationary if the joint cumulative distribution function of any set of samples is invariant with respect to shifts of the origin by integer multiples of some period T_0.

In other words, a process is cyclostationary if for all $t_1, \ldots, t_n \in \mathcal{T}$, any $m \in \mathbb{Z}$ and some $T_0 > 0$,

$$F_{X_{t_1}, X_{t_2}, \cdots, X_{t_n}}(x_1, x_2, \ldots, x_n) = F_{X_{t_1+mT_0}, X_{t_2+mT_0}, \cdots, X_{t_n+mT_0}}(x_1, x_2, \ldots, x_n), \tag{22}$$

for all $x_1, \ldots, x_n \in \mathbb{R}$ (or $\mathbb{C}$ if complex). Such a process is called a **strict-sense cyclostationary (SSCS)** process.

The minimal $T_0 > 0$ for (22) to hold is termed the *period* of the process. We may note that stationary processes are cyclostationary processes for which (22) holds for all $T_0 \in \mathcal{T}$. If $\mathcal{T} = \mathbb{N}$, the minimal period is $T_0 = 1$. If $\mathcal{T} = \mathbb{R}$, there may be no smallest positive period.

The mean and autocorrelation of a cyclostationary second-order process have characteristic periodic forms. For the mean,

$$m_X(t) = \mathbb{E}\left[X_t\right] = \mathbb{E}\left[X_{t+T_0}\right] = m_X(t+T_0), \qquad t \in \mathcal{T}, \tag{23}$$

so that the mean is T_0-periodic: $m_X(t)$ is a periodic function.

For the autocorrelation,

$$\begin{aligned} R_X(t_1,t_2) &= \mathbb{E}\left[X_{t_1}X_{t_2}\right] \\ &= \mathbb{E}\left[X_{t_1+T_0}X_{t_2+T_0}\right] = R_X(t_1+T_0,\, t_2+T_0), \end{aligned} \tag{24}$$

which shows that the autocorrelation is periodic along all lines of constant time difference $t_2 - t_1$.

Definition 3.2. A second-order process $X(t)$, with $\mathcal{T} = \mathbb{R}$ or $\mathcal{T} = \mathbb{Z}$, is called **wide-sense cyclostationary (WSCS)** with period $T_0 > 0$ if

$$m_X(t_1) = m_X(t_1+T_0), \tag{25}$$

$$R_X(t_1,t_2) = R_X(t_1+T_0,\, t_2+T_0), \tag{26}$$

for all $t_1, t_2 \in \mathcal{T}$.

Example 2. Consider a random amplitude sinusoid with period T:

$$X(t) = A\cos\left(\frac{2\pi t}{T}\right).$$

Is $X(t)$ cyclostationary? Wide-sense cyclostationary?

Consider the joint cdf for the time samples $t_1, \ldots, t_k$:

$$\begin{aligned} &P\big[X(t_1) \le x_1,\, X(t_2) \le x_2,\, \ldots,\, X(t_k) \le x_k\big] \\ &\quad = P\big[A\cos(2\pi t_1/T) \le x_1,\, \ldots,\, A\cos(2\pi t_k/T) \le x_k\big] \\ &\quad = P\big[A\cos(2\pi(t_1+mT)/T) \le x_1,\, \ldots,\, A\cos(2\pi(t_k+mT)/T) \le x_k\big] \\ &\quad = P\big[X(t_1+mT) \le x_1,\, X(t_2+mT) \le x_2,\, \ldots,\, X(t_k+mT) \le x_k\big]. \end{aligned}$$

Thus $X(t)$ is a cyclostationary random process and hence also a wide-sense cyclostationary process.

In the above example, the sample functions of the random process are always periodic. The following example shows that, in general, the sample functions of a cyclostationary random process need not be periodic.

Example 3. A modem transmits a binary iid equiprobable data sequence as follows: To transmit a binary 1, the modem transmits a rectangular pulse of duration T seconds and amplitude 1; to transmit a binary 0, it transmits a rectangular pulse of duration T seconds and amplitude -1. Let $X(t)$ be the random process that results. Is $X(t)$ wide-sense cyclostationary?

Let A_n be the sequence of amplitudes (± 1) corresponding to the binary sequence, then $X(t)$ can be represented as the sum of amplitude-modulated time-shifted rectangular pulses:

$$X(t) = \sum_{n=-\infty}^{\infty} A_n p(t-nT). \tag{9.71}$$

The mean of $X(t)$ is

$$\begin{aligned} m_X(t) &= \mathbb{E}\left[\sum_{n=-\infty}^{\infty} A_n p(t-nT)\right] \\ &= \sum_{n=-\infty}^{\infty} \mathbb{E}[A_n]\, p(t-nT) = 0, \end{aligned}$$

since $\mathbb{E}[A_n] = 0$.

The autocovariance function is

$$\begin{aligned} C_X(t_1, t_2) &= \mathbb{E}[X(t_1)X(t_2)] - 0 \\ &= \begin{cases} \mathbb{E}[X(t_1)^2] = 1, & \text{if } nT \le t_1, t_2 < (n+1)T, \\ \mathbb{E}[X(t_1)]\,\mathbb{E}[X(t_2)] = 0, & \text{otherwise.} \end{cases} \end{aligned}$$

It is clear that

$$C_X(t_1 + mT, t_2 + mT) = C_X(t_1, t_2)$$

for all integers m. Therefore the process is wide-sense cyclostationary.